Optimal distributed Kalman filtering for unequal state vectors: Privacy and computational benefits

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Optimal Distributed Kalman Filtering for Unequal State Vectors: Privacy and Computational Benefits

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Abstract—**This paper presents an optimal Kalman filtering (KF) method for distributed systems where agents have unequal state vectors. In contrary to traditional distributed KF methods, the prediction is done centrally at the server. The proposed method allows for an analytically optimal linear estimator that facilitates data transfer efficiency, privacy, and scalability, particularly in scenarios where subsystems have large numbers of measurements. The approach avoids the dimensionality challenges in centralized systems by maintaining local estimates at the distributed agents and minimizing the data transmitted to a central server. The method is evaluated on a spatio-temporal system modeled by a one-dimensional heat equation, subject to homogeneous Dirichlet boundary conditions, and demonstrated to produce near-identical results to centralized Kalman filtering. Applications of the proposed method can be found in distributed multi-target tracking or environmental monitoring. The method shows promise for scenarios where measurements are sensitive or difficult to transfer, providing an optimal and private solution for distributed systems.**

Index Terms—**Distributed Kalman Filter, Unequal state vectors, Distributed estimation, Kalman Filtering**

I. Introduction

In dynamic systems, the choice of estimation method is driven by the system's characteristics. For systems that are linear with Gaussian noise, the centralized Kalman Filter (KF) provides an optimal Bayesian estimate [1]. The centralized KF assumes a centralized system that has access to all measured data. If the data is collected in a distributed manner, this may not be possible in a practical setting. Accessing all data can be problematic due to privacy concerns, highlighted in Federated Learning (FL) [2], and communication limitations [3], [4]. An example is a vehicle target tracking scenario where the states of a vehicle are estimated both by an external estimator and locally by the vehicle itself. To construct a centralized KF, the external estimator would need access to all the measurements used in the local estimator. These measurements could consist of a combination of RADAR, LiDAR and GPS, which may contain a large amount of data. Additionally, there may be privacy concerns. The vehicle might not want to share these raw measurements with the external estimator.

An alternative to centralized estimation is Distributed Data Fusion (DDF). In DDF, each agent estimates the system states individually, which are then fused to obtain a more accurate estimate than any individual estimate. DDF and its applications are widely studied [5], [6]. In [7], the Bar-Shalom/Campo (BC) fuser was developed, which is an optimal way of fusing two estimates of the same state. A limitation of the BC fuser is that it requires knowledge of the cross-covariance between estimates, which is rarely available in practical applications. If the cross covariance is (wrongly) assumed to be zero, the resulting fusion is called naive. Naive fusion typically underestimates the covariance of the fused estimate due to double counting data, as the two estimates usually share common information since they are estimates of the same physical process [8], [9]. Covariance Intersection (CI) [10] offers an alternative to naive fusion, it provides conservative fusion irrespective of cross covariances, meaning that it does not underestimate the covariance of the fused estimate. Multiple alternatives and derivatives of CI exist, such as the Federated Kalman Filter [11], the Largest Ellipsoid method [12], and the split CI [13]. CI and its derivatives perform well when all distributed agents estimate the same state. However, CI cannot be directly applied when distributed agents estimate different subspaces of a full global state, a scenario known as distributed estimation of unequal states. Distributed estimation of unequal states has previously been studied in [14]–[17]. In [17], different methods of augmenting the partial states before fusion are investigated. In [14], [16] a fusion method for unequal states by formulating the fusion as a Weighted Least Squares problem is proposed, including an adaptation to CI for unequal states if the cross covariances are unknown. Adapting CI to unequal states guarantees conservativeness, although the fused estimate typically becomes excessively conservative.

In this paper, we propose a novel strategy for distributed estimation of unequal state vectors by using a global prior for the distributed agents. This differs from the typical setup seen in literature where priors are local to each agent. To the best of the authors knowledge, this has not previously been considered. The main contributions are:

1) We propose a novel distributed Kalman filter, that maintains a common global prior prediction of the latent states, while performing distributed measurement updates on local reduced state subsystems.
2) We use the method to fuse the distributed estimates into a global state using only the local state estimates and covariance matrices, without sharing the raw measurements, measurement covariance or observation matrices.

This maintains the privacy of the measurements and if the distributed agents have more measurements than states, it also

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reduces the complexity of centralized computations.

II. Problem formulation

This paper considers the linear estimation of a global state using local distributed estimates of its subspaces. The global system is characterized by the discrete time process

$$x_n^* = \mathbf{A}_n x_{n-1}^* + w_n^* \tag{1}$$

where $\mathbf{A}_n \in \mathbb{R}^{d_{x^*} \times d_{x^*}}$, $x_n^* \in \mathbb{R}^{d_{x^*}}$ and $w_n^* \in \mathbb{R}^{d_{x^*}}$ are the system matrix, the state vector, and zero mean white Gaussian process noise with covariance matrix $\mathbf{Q}_n^*$ at time n, respectively. The superscript x^* denotes the global state vector. Each distributed subsystem state is given by

$$x_n^l = \mathbf{M}_n^l x_n^* \qquad l = 1, \ldots, L \tag{2}$$

where $x_n^l \in \mathbb{R}^{d_{x^l}}$, denotes the lth local state and $\mathbf{M}_n^l \in \mathbb{R}^{d_{x^l} \times d_{x^*}}$ is a mapping from the global state to the subsystem.

Each agent measures an arbitrary subset of the local states

$$y_n^l = \mathbf{H}_n^l x_n^l + v_n^l \tag{3}$$

where $y_n^l \in \mathbb{R}^{d_{y^l}}$ is the measurement, $\mathbf{H}_n^l \in \mathbb{R}^{d_{y^l} \times d_{x^l}}$ is the observation matrix, and $v_n^l \in \mathbb{R}^{d_{y^l}}$ is zero mean white Gaussian measurement noise with covariance matrix $\mathbf{R}_n^l \in \mathbb{R}^{d_{y^l} \times d_{y^l}}$. It is assumed that there is no relation between measurements in different subsystems, which means that local subsystems only measure their own states and $\mathrm{Cov}(v_n^i, v_n^j) = 0, \ \forall \ i \neq j$.

Assuming that all measurement data with corresponding covariance matrix

$$y_n^* = \begin{bmatrix} y_n^1 \\ \vdots \\ y_n^L \end{bmatrix}, \quad \mathbf{R}_n = \begin{bmatrix} \mathbf{R}_n^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{R}_n^L \end{bmatrix}$$

and global observation matrix $\mathbf{H}_n^* \in \mathbb{R}^{d_{y^*} \times d_{x^*}}$, are available at a centralized server, the optimal estimate of the global system (1), is given by the centralized KF given by [18]

1. Prediction:

$$\hat{x}_{n|n-1}^* = \mathbf{A}_n \, \hat{x}_{n-1|n-1}^* \tag{4a}$$
$$\mathbf{P}_{n|n-1}^* = \mathbf{A}_n \, \mathbf{P}_{n-1|n-1}^* \, \mathbf{A}_n^\intercal + \mathbf{Q}_n^* \tag{4b}$$

2. Measurement update:

$$z_n^* = y_n^* - \mathbf{H}_n^* \, \hat{x}_{n|n-1}^* \tag{5a}$$
$$\mathbf{S}_n^* = \mathbf{H}_n^* \, \mathbf{P}_{n|n-1}^* \, \mathbf{H}_n^{*\intercal} + \mathbf{R}_n^* \tag{5b}$$
$$\mathbf{K}_n^* = \mathbf{P}_{n|n-1}^* \, \mathbf{H}_n^{*\intercal} \, (\mathbf{S}_n^*)^{-1} \tag{5c}$$
$$\hat{x}_{n|n}^* = \hat{x}_{n|n-1}^* + \mathbf{K}_n^* \, z_n^* \tag{5d}$$
$$\mathbf{P}_{n|n}^* = (\mathbf{I} - \mathbf{K}_n^* \, \mathbf{H}_n^*) \, \mathbf{P}_{n|n-1}^* \tag{5e}$$

where $\hat{x}_{n|n}^*$ indicates the estimate of x_n^* based on all measurements up to and including time n. The global Kalman filter will provide an optimal estimate, but there are practical and privacy concerning challenges regarding the measurements of the subsystems. Transferring the raw data may require using a lot of bandwidth, and the distributed agents may not want to share their data with other agents or the centralized server.

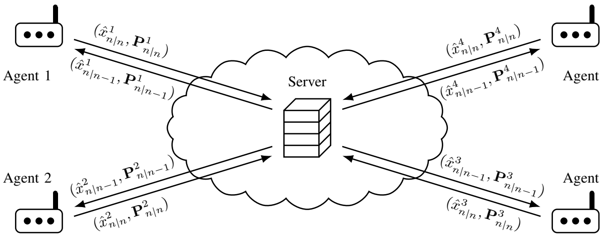

Fig. 1. The considered setup with $L = 4$ distributed agents. Each distributed agent estimates a subspace of the global state, using the marginalization $(\hat{x}_{n|n-1}^l, \mathbf{P}_{n|n-1}^l)$ given by (7). The server fuses the local estimates and predicts the next state using the system dynamics. The communication, prediction, updates and fusion occurs during each time step.

Hence, we consider the case where each agent performs a local measurement update

$$\mathbf{K}_n^l = \mathbf{P}_{n|n-1}^l \mathbf{H}^{l,\intercal} (\mathbf{H}^l \mathbf{P}_{n|n-1}^l \mathbf{H}^{l,\intercal} + \mathbf{R}_n^l)^{-1} \tag{6a}$$
$$\hat{x}_{n|n}^l = \hat{x}_{n|n-1}^l + \mathbf{K}_n^l (y_n^l - \mathbf{H}^l \hat{x}_{n|n-1}^l) \tag{6b}$$
$$\mathbf{P}_{n|n}^l = (\mathbf{I} - \mathbf{K}_n^l \mathbf{H}^l) \mathbf{P}_{n|n-1}^l, \tag{6c}$$

based on common prior information obtained from the global system, which is maintained by a central server. The central server performs the global prediction (4), and transfers the (potentially) reduced prior state and covariance estimates to each respective agent according to

$$\hat{x}_{n|n-1}^l = \mathbf{M}_n^l \, \hat{x}_{n|n-1}^* \tag{7a}$$
$$\mathbf{P}_{n|n-1}^l = \mathbf{M}_n^l \, \mathbf{P}_{n|n-1}^* \, (\mathbf{M}_n^l)^\intercal. \tag{7b}$$

The updated reduced-state estimates and covariances are then transferred back to the centralized server to perform a global update without needing the local measurements, measurement covariances or observation matrices. An overview of the setup is illustrated in Fig. 1.

III. Global update from local estimate

A. Equivalent measurements and covariances

In the standard KF measurement update (5), each measurement in y_n^* updates all states in the global system proportional to the predicted cross covariance with the measured state. If the global system (1) is expanded such that additional states are introduced, and the cross covariance between the newly introduced states and the rest of the state vector is known, the only part of (5c) that changes is the matrix product $\mathbf{P}_{n|n-1}^* \mathbf{H}_n^{*\intercal}$. The only change that needs to be made to include the newly added states in the update, is adding the new states into $\mathbf{P}_{n|n-1}^*$ to form the augmented covariance matrix $\mathbf{P}_{n|n-1}^{aug}$, as well as adding a zero column to $\mathbf{H}_n^*$ corresponding to each added state, forming the augmented observation matrix $\mathbf{H}_n^{aug}$.

By utilizing the fact that the global state in (4) has a common prior distribution and that the locally calculated estimates are available, it is possible to augment all distributed subspace estimates to obtain L global estimates if the structure of the observation matrices $\mathbf{H}_n^l$ is known. This is in fact still possible to do without knowing $\mathbf{H}_n^l$ and thus, without compromising the integrity of the distributed agents.

Consider the local measurement update of the lth agent, which consists of d_{x^l} states, with d_{y^l} observations that span a subspace D^l, where $\dim(D^l) = \lambda^l \leq d_{x^l}$. Using (5e), it is possible to calculate

$$\mathbf{K}_n^l \mathbf{H}_n^l = \big(\mathbf{P}_{n|n-1}^l - \mathbf{P}_{n|n}^l\big)\big(\mathbf{P}_{n|n-1}^l\big)^{-1} \tag{8}$$

and determine the rank of the matrix product $\mathbf{K}_n^l \mathbf{H}_n^l$, which is λ^l. Next, let there exist a matrix product

$$\bar{\mathbf{K}}_n^l \bar{\mathbf{H}}_n^l = \mathbf{K}_n^l \mathbf{H}_n^l, \tag{9}$$

where the factors $\bar{\mathbf{K}}_n^l$, and $\bar{\mathbf{H}}_n^l$ are initially unknown. Now, assign $\bar{\mathbf{H}}_n^l \in \mathbb{R}^{\lambda^l \times d_{x^l}}$ to be a matrix such that

$$\bar{\mathbf{K}}_n^l = \big(\mathbf{K}_n^l \mathbf{H}_n^l\big)\big(\bar{\mathbf{H}}_n^l\big)^{\dagger}, \tag{10}$$

extracts the linearly independent columns of $\mathbf{K}_n^l \mathbf{H}_n^l$, where † denotes the pseudo-inverse. In the case that $\mathbf{K}_n^l \mathbf{H}_n^l$ is full rank, the matrix $\bar{\mathbf{H}}_n^l$ will simply be the identity matrix, even if $d_{y^l} > d_{x^l}$. If $\mathbf{K}_n^l \mathbf{H}_n^l$ is rank deficient, the rows of $\bar{\mathbf{H}}_n^l$ are given by the right singular vectors, of $\mathbf{K}_n^l \mathbf{H}_n^l$, corresponding to the non-zero singular values.

Given that the structure of $\bar{\mathbf{K}}_n^l$ is known from (5b) and (5c), it is possible to calculate a transformed, but equivalent, measurement covariance $\bar{\mathbf{R}}_n^l$, and measurement $\bar{y}_n^l$ as

$$\bar{\mathbf{R}}_n^l = \Big((\bar{\mathbf{H}}_n^{l\intercal})^\dagger (\mathbf{P}_{n|n-1}^l)^{-1} \bar{\mathbf{K}}_n^l\Big)^{-1} - \bar{\mathbf{H}}_n^l \mathbf{P}_{n|n-1}^l \bar{\mathbf{H}}_n^{l\intercal} \tag{11a}$$

$$\bar{y}_n^l = (\bar{\mathbf{K}}_n^l)^\dagger \Big(\hat{x}_{n|n}^l + (\bar{\mathbf{K}}_n^l \bar{\mathbf{H}}_n^l - \mathbf{I})\hat{x}_{n|n-1}^l\Big) \tag{11b}$$

This leads to the transformed KF update

$$\bar{z}_n^l = \bar{y}_n^l - \bar{\mathbf{H}}_n^l \, \hat{x}_{n|n-1}^l \tag{12a}$$

$$\bar{\mathbf{S}}_n^l = \bar{\mathbf{H}}_n^l \, \mathbf{P}_{n|n-1}^l \, \bar{\mathbf{H}}_n^{l\intercal} + \bar{\mathbf{R}}_n^l \tag{12b}$$

$$\bar{\mathbf{K}}_n^l \, \bar{z}_n^l = \mathbf{P}_{n|n-1}^l \, \bar{\mathbf{H}}_n^{l\intercal} \big(\bar{\mathbf{S}}_n^l\big)^{-1} \bar{z}_n^l \tag{12c}$$

$$= \mathbf{K}_n^l \, z_n^l \tag{12d}$$

which is equivalent to the update in (5d) if the reduced prior is replaced with the global and the observation matrices are augmented. This means that the computed $\bar{y}_n^l$ and $\bar{\mathbf{R}}_n^l$ are linear transformations of the original y_n^l and $\mathbf{R}_n^l$. This can easily be seen if $\lambda^l = d_{y^l} = d_{x^l}$: In this case

$$\bar{\mathbf{H}}_n^l = (\mathbf{H}_n^l)^{-1} \mathbf{H}_n^l = \mathbf{I} \tag{13a}$$

$$\bar{y}_n^l = (\mathbf{H}_n^l)^{-1} y_n^l \tag{13b}$$

$$\bar{\mathbf{R}}_n^l = (\mathbf{H}_n^l)^{-1} \mathbf{R}_n^l \Big((\mathbf{H}_n^l)^{-1}\Big)^\intercal \tag{13c}$$

Since the measurements of the distributed agents are assumed to be independent of each other, the result in (12) can be expanded to include all distributed agents and obtain an equivalent full state KF update.

B. Proposed full state update

By augmenting the observation matrices with zeros for the non-common states between the distributed agents and the global system, and using the common prior $\mathbf{P}_{n|n-1}^*$, we can obtain L global estimates $(\hat{x}_{n|n}^{*,l}, \mathbf{P}_{n|n}^{*,l})$, for $l = 1, \ldots, L$. $(\hat{x}_{n|n}^{*,l}, \mathbf{P}_{n|n}^{*,l})$ is an estimate of the global state vector from the lth agent's local estimate, a projection from the subspace to the global state vector. All L projections could now be fused using state of the art methods such as CI [10].

However, we instead propose an optimal KF update using the calculated $\bar{y}_n^l$ and $\bar{\mathbf{R}}_n^l$. Consider

$$\bar{y}_n^* = \begin{bmatrix} \bar{y}_n^1 \\ \vdots \\ \bar{y}_n^L \end{bmatrix}, \bar{\mathbf{R}}_n^* = \begin{bmatrix} \bar{\mathbf{R}}_n^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{\mathbf{R}}_n^L \end{bmatrix}, \bar{\mathbf{H}}_n^* = \begin{bmatrix} \bar{\mathbf{H}}_n^{1,\ aug} \\ \vdots \\ \bar{\mathbf{H}}_n^{L,\ aug} \end{bmatrix},$$

which are the equivalent measurement vector, noise covariance, and the joint estimated observation matrix, respectively. These can be used for the equivalent global KF update

$$\bar{\mathbf{K}}_n^* = \mathbf{P}_{n|n-1}^* \, \bar{\mathbf{H}}_n^{*\ \intercal} \Big(\bar{\mathbf{H}}_n^* \, \mathbf{P}_{n|n-1}^* \, \bar{\mathbf{H}}_n^{*\ \intercal} + \bar{\mathbf{R}}_n^*\Big)^{-1}, \tag{14a}$$

$$\hat{x}_{n|n}^* = \hat{x}_{n|n-1}^* + \bar{\mathbf{K}}_n^* \Big(\bar{y}_n - \bar{\mathbf{H}}_n^* \, \hat{x}_{n|n-1}^*\Big), \tag{14b}$$

$$\mathbf{P}_{n|n}^* = \Big(\mathbf{I} - \bar{\mathbf{K}}_n^* \, \bar{\mathbf{H}}_n^*\Big) \, \mathbf{P}_{n|n-1}^*. \tag{14c}$$

The proposed method is summarized in Algorithm 1.

Algorithm 1 Fusion of distributed unequal state vectors

1: Kalman prediction with the global state vector and covariance matrix (4).
2: **for** $l = 1, 2, \ldots, L$ **do**
3: Extract and transfer the local state predictions (7).
4: Perform local update (6) and transfer to the server.
5: **end for**
6: **for** $l = 1, 2, \ldots, L$ **do**
7: Receive local update $\hat{x}_{n|n}^l$ and $\mathbf{P}_{n|n}^l$.
8: Determine $\bar{\mathbf{H}}_n^l$ from (9).
9: Compute $\bar{y}_n^l$, $\bar{\mathbf{R}}_n^l$ from (11).
10: **end for**
11: Construct $\bar{y}_n^*$, $\bar{\mathbf{R}}_n^*$ and $\bar{\mathbf{H}}_n^*$.
12: Perform measurement update (14) to obtain $\hat{x}_{n|n}^*$ and $\mathbf{P}_{n|n}^*$.

C. Proof of equivalence in a special case

Consider the regular KF update with access to the exact measurements y_n^*, measurement covariance $\mathbf{R}_n^*$ and observation matrices

$$\mathbf{H}_n^* = \begin{bmatrix} \mathbf{H}_n^{1,\ aug} \\ \vdots \\ \mathbf{H}_n^{L,\ aug} \end{bmatrix} \tag{15}$$

from the distributed agents. Furthermore, assume that the total number of observations equals the number of total states such that $\mathbf{K}_n^* \mathbf{H}_n^*$ is full rank, and thus, $\mathbf{H}_n^*$ is square. By transforming y_n^*, $\mathbf{R}_n^*$ and $\mathbf{H}_n^*$ with an arbitrary non-singular weight matrix $\mathbf{W}$ according to (13), we can see that

$$\begin{aligned}
&\mathbf{P}^*_{n|n-1}\,\mathbf{H}_n^\intercal\,\mathbf{W}^\intercal\Big(\mathbf{W}\big(\mathbf{S}^*_\mathbf{n}\big)\mathbf{W}^\intercal\Big)^{-1}\mathbf{W}\,z_n^*\\
&=\mathbf{P}^*_{n|n-1}\,\mathbf{H}_n^\intercal\,\mathbf{W}^\intercal\Big((\mathbf{W}^\intercal)^{-1}\big(\mathbf{S}^*_\mathbf{n}\big)^{-1}\mathbf{W}^{-1}\Big)\mathbf{W}\,z_n^*\\
&=\mathbf{P}^*_{n|n-1}\,\mathbf{H}_n^\intercal\big(\mathbf{S}^*_\mathbf{n}\big)^{-1}\Big(y_n^*-\mathbf{H}_n\,\hat{x}^*_{n|n-1}\Big),
\end{aligned}$$

which shows that the transformed KF update is equivalent to the regular KF update (5).

IV. Results

We evaluated the proposed method on two examples and compare it with the centralized KF using the RMS relative error (RMSRE) [19]

$$\text{RMSRE}(i)=\sqrt{\frac{1}{N}\sum_{n=1}^{N}\frac{||\hat{x}^{prop}_{n|n}-\hat{x}^{KF}_{n|n}||^2}{||\hat{x}^{KF}_{n|n}||^2}}. \tag{16}$$

A. Toy example

In order to demonstrate the feasibility of the proposed method, we show a toy example for one time step. Consider the global system

$$x_n=\frac{1}{100}\begin{bmatrix}82&8&0&0\\8&75&7&0\\0&7&68&6\\0&0&6&61\end{bmatrix}x_{n-1}+w_n,\, w_n\sim\mathcal{N}\left(\mathbf{0},\frac{1}{100}\begin{bmatrix}8&1&0&0\\1&8&1&0\\0&1&7&1\\0&0&1&6\end{bmatrix}\right),$$

that contains 4 states, and is divided up into 2 subsystems, $\mathbf{A}$ and $\mathbf{B}$, each containing 2 substates of the global states.

A centralized server runs a Kalman prediction of the global state (4), and transfers marginalizations of the global state (7) to $\mathbf{A}$ and $\mathbf{B}$. Two cases are considered for this example:

Case 1: Subsystems $\mathbf{A}$ and $\mathbf{B}$ each measure their respective states using a total of 3 observations each, with their respective observations matrices

$$\mathbf{H}^\mathbf{A}=\begin{bmatrix}1&0.3\\2&2\\0.3&1\end{bmatrix},\mathbf{H}^\mathbf{B}=\begin{bmatrix}2&1\\1&3\\0.3&0.5\end{bmatrix}.$$

For comparison, a centralized KF is used to estimate the system, using the global observation matrix

$$\mathbf{H}^*=\begin{bmatrix}\mathbf{H}^\mathbf{A}&\mathbf{0}\\\mathbf{0}&\mathbf{H}^\mathbf{B}\end{bmatrix}\in\mathbb{R}^{6\times4}. \tag{17}$$

Case 2: Subsystems $\mathbf{A}$ and $\mathbf{B}$ each measure their respective states using a total of 1 observation each, with their respective observations matrices

$$\mathbf{H}^\mathbf{A}=\begin{bmatrix}1&0.3\end{bmatrix},\mathbf{H}^\mathbf{B}=\begin{bmatrix}2&1\end{bmatrix}.$$

For comparison, a centralized KF is used to estimate the system, using the global observation matrix

$$\mathbf{H}^*=\begin{bmatrix}\mathbf{H}^\mathbf{A}&\mathbf{0}\\\mathbf{0}&\mathbf{H}^\mathbf{B}\end{bmatrix}\in\mathbb{R}^{2\times4}. \tag{18}$$

Table I presents the RMSRE (16) between the centralized KF and the method described in Algorithm 1, for both test cases. The results highlight the advantages of the proposed method in terms of privacy. Neither the true measurement, nor the measurement covariance, nor the number of measurements are revealed at the server. Case 1 also highlights the benefits in terms of communication and server side computation. If the number of measurements increase, the calculated observation matrix $\bar{\mathbf{H}}^*$ will still be a $\mathbf{I}_{4\times4}$.

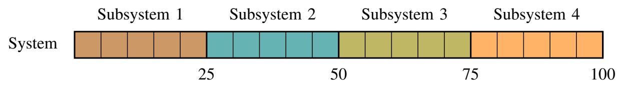

Fig. 2. The setup considered for all test cases (a), (b), and (c). The global system consist of 100 states, and each subsystem consists of 25 non-overlapping states.

B. The one dimensional heat equation

In the second example, we consider a one dimensional heat equation subject to homogeneous Dirichlet boundary conditions,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+w(x,t), \tag{19}$$

where $w(x,t)$ is a white noise process with spectral density σ^2. The system (19) is spatially discretized into 100 equally spaced sections, one meter apart, using the finite difference method, and temporally discretized [20] to yield the global system described in (1). An overview is shown in Fig. 2.

Three test cases are considered. In all cases, the system is divided up into four equally sized subsystems, each containing 25 states with no overlap. The process noise is Independent and Identically Distributed (IID) White Gaussian Noise (WGN) with $\sigma_p^2=2$. The sampling time is $\Delta t=0.1\,\text{s}$ and the measurements are subject to IID WGN with $\sigma_m^2=1$. Each case is performed for $I=500$ Monte Carlo (MC) runs, with $N=100$ iterations in each run. At the end of each MC run, the proposed method is compared to a fully centralized KF using the RMSRE (16). The specific test cases are:

a) Each subsystem contains 25 local states and measures 15 of the local states, with a total of $d_{y^l}=15$ observations. During each MC run, a new subset of 15 states is selected and the observation matrices $\mathbf{H}^l$ are randomized for these specific states. The observation matrices remain constant during each iteration of the same MC run.
b) Each subsystem contains 25 local states and measures all local states, with a total of $d_{y^l}=15$ observations. During each MC run, the observation matrices $\mathbf{H}^l$ are randomized. The observation matrices remain constant during each iteration of the same MC run.
c) Each subsystem contains 25 local states and measures all 25 local states, with a total of $d_{y^l}=35$ observations. During each MC run, the observation matrices $\mathbf{H}^l$ are randomized. The observation matrices remain constant during each iteration of the same MC run.

For the one dimensional heat equation we compare the proposed method against a centralized KF by evaluating the RMSRE (16) in all three test cases. As the proposed method is analytically equivalent to the centralized KF, this is the only comparison of interest. The means and variances of the RMSRE are

TABLE I
TOY EXAMPLE RESULTS FOR BOTH TEST CASES

Case	$\mathbf{H}^*$	$\bar{\mathbf{H}}^*$	y	$\bar{y}$	$\mathbf{R}$	$\bar{\mathbf{R}}$	RMSRE
1	$\begin{bmatrix} 1.0 & 0.3 & 0 & 0 \\ 2.0 & 2.0 & 0 & 0 \\ 0.3 & 1.0 & 0 & 0 \\ 0 & 0 & 2.0 & 1.0 \\ 0 & 0 & 1.0 & 3.0 \\ 0 & 0 & 0.3 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.10 \\ 0.06 \\ 0.76 \\ -0.72 \\ -0.83 \\ -0.90 \end{bmatrix}$	$\begin{bmatrix} -0.56 \\ 0.67 \\ -0.29 \\ -0.21 \end{bmatrix}$	$0.5\mathbf{I}_{6\times 6}$	$\begin{bmatrix} 0.54 & -0.48 & 0 & 0 \\ -0.48 & 0.54 & 0 & 0 \\ 0 & 0 & 0.20 & -0.10 \\ 0 & 0 & -0.10 & 0.10 \end{bmatrix}$	4.37×10^{-16}
2	$\begin{bmatrix} 1.0 & 0.3 & 0 & 0 \\ 0 & 0 & 2.0 & 1.0 \end{bmatrix}$	$\begin{bmatrix} -0.96 & -0.29 & 0 & 0 \\ 0 & 0 & -0.89 & -0.45 \end{bmatrix}$	$\begin{bmatrix} -0.10 \\ -0.72 \end{bmatrix}$	$\begin{bmatrix} 0.10 \\ 0.33 \end{bmatrix}$	$0.5\mathbf{I}_{2\times 2}$	$\begin{bmatrix} 0.46 & 0 \\ 0 & 0.10 \end{bmatrix}$	1.52×10^{-16}

summarized in Table II. The difference of the methods are on the order of magnitude of machine precision.

TABLE II
PERFORMANCE METRICS OF DIFFERENT TEST CASES

Case	Mean	Variance
(a)	1.24×10^{-14}	6.89×10^{-30}
(b)	6.40×10^{-15}	1.28×10^{-30}
(c)	4.77×10^{-15}	3.14×10^{-31}

V. CONCLUSIONS

The proposed method demonstrates near identical results to the centralized KF in simulations, with the main differences being in the order of magnitude of machine precision. The proposed method allows for an analytically optimal linear estimator for distributed systems of unequal states, which has advantages over the centralized KF in terms of data transfer, privacy and computational time, particularly when the subsystems have a large number of measurements. Even if the number of measurements far exceeds the number of states at the local agent, the calculations done at the central server assume that the number of measurements equals the number of states. This can significantly reduce the dimensionality of the matrix inversions at the centralized server. In addition to this, since only the state estimates and covariance matrices are sent back to the centralized server, the integrity of the distributed agents is not compromised.

These results can be particularly useful in environmental monitoring, as demonstrated in this paper, or distributed target tracking scenarios. For example on an accident-prone stretch of road, where a central server could be placed to estimate all states of the vehicles on this stretch. Modern vehicles have access to a lot of measurements and data, some of which may be private or unfeasible to transfer. In this scenario, the proposed method could provide the server with an efficient optimal estimate of the system, while still preserving the privacy.

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