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Vehicle Tracking Based on Fusion of Magnetometer and Accelerometer Sensor Measurements with Particle Filtering

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Abstract-In this article, we propose a method for vehicle tracking on roadways using measurements of magnetometers and accelerometers. The measurements are used to build a low-cost, low-complexity vehicle tracking sensor platform for highway traffic monitoring. First, the problem is formulated by introducing the process model for the motion of the vehicle on the road and two measurement models, one for each of the sensors. Second, it is shown how the measurements of the sensors can be fused using particle filtering. The standard sampling importance resampling (SIR) particle filter is extended for processing of multi-rate sensor measurements and models that employ unknown static parameters. The latter are treated by Rao-Blackwellization. The performance of the method is demonstrated by computer simulations. It is found that it is feasible to fuse the two sensors for vehicle tracking and that the proposed multi-rate particle filter performs better than particle filters that process only measurements of one of the sensors. The main contribution of this paper is the novel approach of fusing the measurements of road-mounted magnetometers and accelerometers for vehicle tracking and traffic monitoring.

Index Terms-Particle filters, sensor fusion, vehicle tracking

I. INTRODUCTION

Target tracking is of importance in many different applications ranging from air traffic control [2] to tracking of mobile phone users within a cellular network [3]. Another important application is in intelligent transportation systems that allows for tracking of vehicles on roadways. This is of interest for obtaining insightful information about the traffic which can be used, for example for understanding traffic patterns such as congestions [4], [5] or for predicting/preventing accidents [6] (we note that the prevention of accidents requires very high accuracy of tracking). The gathered information (aggregate or individual) can be broadcast to individual vehicles equipped with corresponding receiver equipment [4].

Popular approaches for vehicle tracking itself are often based on vision systems [7]-[10], radar [11], or a combination of such techniques [12]. Recently, solutions that employ low-cost, low-complexity sensors such as microphones [10], [13], magnetometers [14], [15], or accelerometers [16] have emerged.

R. Hostettler is with the Department of Computer Science, Electrical and Space Engineering, Luleå University of Technology, Luleå, Sweden (e-mail: roland.hostettler@ltu.se). Target tracking is often formulated as a state estimation problem where the position of the target as a function of time is considered a random process [17]. The measurements obtained from the sensors are described as a function of the states, for example, of the range and bearing of a target [18]. In many cases, these measurements are highly nonlinear functions of the states which, for their processing, often require the use of approximating methods, such as the extended Kalman or particle filters. It has been shown that the latter is a suitable approach in many different applications; see [2] for a thorough overview.

In this work, we address vehicle tracking by combining magnetometer and accelerometer measurements in a single sensor unit that is mounted on the road surface as illustrated in [16]. Even though tracking using individual sensors has been addressed before [14]-[16], the combination of the two sensors has not been considered yet. An obvious major advantage is that the two different sensors come at low costs and that they complement each other in the sense that they measure completely different phenomena. Furthermore, the sensors can be integrated in small, battery-powered sensor nodes and require less computational power than, for example, video-based systems.

Magnetometers are sensors measuring the strength of the Earth's magnetic field at a given point. They are often used in compass applications, for example, in mobile phones. Metallic objects like vehicles cause local distortions, and they can be measured by a magnetometer and exploited for target tracking [19]. Commonly, magnetometers are vector sensors measuring the Cartesian components of the vector field. Accelerometers are also used in a number of applications. An accelerometer attached to the road surface measures vibrations in the road caused by dynamic loads of vehicles passing in its proximity. The features of the vibrations can be used to estimate vehicle parameters.

When these sensors are used for tracking purposes, there are several important issues that need to be addressed carefully. First, the sensor measurements are highly nonlinear functions of the states. Second, the measurements depend on a set of unknown, target- or material-specific parameters which, if included in the estimation problem, make the problem even more challenging. Third, due to the different measurement principles, the sensors have different properties including different sampling rates and operating clocks that usually are not synchronized.

The solutions to this type of problems require fusion of

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information from the sensors and multi-rate processing. A methodology that is suitable for these problems is particle filtering. Some solutions for generically similar settings have been proposed in the literature. For example, a hybrid multirate Kalman-particle filter was proposed in [20]. There, it was assumed that the measurements for the faster sensor depend linearly on the state and hence, a Kalman filter was used to update the particles when only the linear measurement were available. The method was also applied to a tracking problem where velocity and range measurements were available. In [21], particle filtering with multi-rate sensors in a process industry context was considered. Similarly, [22] introduced multi-rate particle filtering in conjunction with robot localization based on a set of different sensors. An unscented information filter for estimating the position of a vehicle fusing the measurements from a camera, laser range finder, and GPS receiver was proposed in [23]. Finally, there is a large body of work on vehicle tracking using particle filtering and different types of sensors. For example, Yin et. al. propose to combine a particle filter with the CamShift algorithm for vehicle tracking using video in order to achieve scale-invariability and account for disturbances such as occlusion and background clutter [24]. Another video-based method that is especially robust to partial occlusion of the target was proposed in [25]. Finally, [26] also uses particle filtering for video-based vehicle tracking where particles are clustered by analyzing the motion coherence in order to form convex shapes of the tracked objects.

The contributions of this paper are as follows. First, we propose a multi-rate particle filtering method for vehicle tracking fusing the measurements of two passive sensors, a magnetometer and an accelerometer, where the measurements are in general asynchronous. This is a novel approach and has not been considered before. Second, the unknown parameters in the sensor models are handled through Rao-Blackwellization, simplifying the tracking problem and improving the performance. With our simulation results, we show that the proposed method can be used for implementing a lowcost, high accuracy vehicle tracking sensor platform.

The remainder of this paper is organized as follows. The problem is formulated in Section II and the particle filtering algorithm is introduced in Section III. How the particle filter can be applied to the problem is shown in Section IV. Finally, simulation results are provided in Section V and concluding remarks are given in Section VI.

II. PROBLEM FORMULATION

We formulate the problem by using a model describing the vehicle motion along the road and two measurement models, for the magnetometer and accelerometer, respectively. They are described in turn below.

A. Motion Model

The vehicle is assumed to follow the course of the road, which is a very constrained environment. Furthermore, since the time window where the vehicle is in the vicinity of the sensor is rather short (of magnitude $1 \le ... 2 \le$), it can be safely assumed that the vehicle travels at a nearly constant speed.

Thus, the motion is represented by a simple one-dimensional constant velocity motion model given by

$$x_{t} = \underbrace{\begin{bmatrix} 1 & T_{s} \\ 0 & 1 \end{bmatrix}}_{\triangleq A} x_{t-1} + \underbrace{\begin{bmatrix} \frac{T_{s}^{2}}{2} \\ T_{s} \end{bmatrix}}_{\triangleq B} u_{t}, \tag{1}$$

where t = 1, 2, ..., T is a discrete time index, T_s is the sampling time,

$$x_t = \begin{bmatrix} r_t^x \\ \dot{r}_t^x \end{bmatrix},\tag{2}$$

 r_t^x is the position on the road, and \dot{r}_t^x is the vehicle's velocity. The symbol u_t can be interpreted as a random acceleration term that accounts for small changes in the vehicles velocity due to the drivers input, drag, and so on. It is a random variable with a distribution given by

$$p(u_t) = \mathcal{N}(u_t; 0, \sigma_{u_t}^2), \tag{3}$$

where $\mathcal{N}(u_t; 0, \sigma_{u_t}^2)$ signifies normal distribution of a scalar with mean zero and variance $\sigma_{u_t}^2$.

The initial position of the vehicle is the point where the vehicle is in the proximity of the sensor and where the measurements start being taken. It is thus given through the sensor range plus some uncertainty. Hence, it is modeled according to

$$p(x_0) = \mathcal{N}(x_0; \mu_{x_0}, C_{x_0}), \tag{4}$$

where μ_{x_0} is the mean sensor range, C_{x_0} the spread, and

$$\mathcal{N}(x_0; \mu_{x_0}, C_{x_0}) = \frac{1}{(2\pi)^{N/2} |C_{x_0}|^{1/2}} \times e^{-\frac{1}{2}(x_0 - \mu_{x_0})^\top C_{x_0}^{-1}(x_0 - \mu_{x_0})}$$
(5)

is the normal distribution of an $N \times 1$ random variable x_0 with mean μ_{x_0} and covariance C_{x_0} .

Finally, the position of the target is defined by the vector

$$r_t = \begin{bmatrix} r_t^x \\ r_y^y \\ r_t^z \end{bmatrix}$$
(6)

pointing from the sensor at the origin of the Cartesian coordinate system to the target. The position r_t^z in the z-direction is assumed to be $r_t^z = 0$, that is, it is assumed that the sensor is in the same plane as the vehicle. Furthermore, the lateral position r_t^y is assumed to be in the middle of the driving lane adjacent to the sensor and hence known.

B. Magnetometer

The magnetic mass of the vehicle causes small local magnetic distortions in the earth's magnetic field in the vicinity of the vehicle which can be measured using a magnetometer [15], [19]. It has been shown and verified [15] that the measured magnetic field can be modeled by the static background field B_0 plus a series of dipoles with locations r_p and magnetic moments m_p as

$$B = B_0 + \sum_{p=1}^{P} \frac{3(r_p^{\top} m_p)r_p - |r_p|^2 m_p}{|r_p|^5}.$$
 (7)

The model order P depends on the magnetic extension of the vehicle as well as the distance between the vehicle and the sensor. It has been shown [15] that a low model order of P = 1...3 is sufficient for small- to mid-sized vehicles passing the sensor at a reasonable distance. For simplicity, we assume a single dipole (P = 1) here, however, the extension to higher orders is straightforward. Furthermore, the static background field B_0 can be subtracted which finally yields the measurement model

$$y_{m,t} = \underbrace{\frac{3(r_t^{\top}m)r_t - |r_t|^2m}{|r_t|^5}}_{\triangleq h_m(x_t)} + w_{m,t},$$
(8)

where $y_{m,t}$ is a 3×1 vector containing the vector components of the magnetic field, $w_{m,t}$ is the measurement noise with a distribution given by

$$p(w_{m,t}) = \mathcal{N}(w_{m,t}; 0, \sigma_m^2 I_3), \tag{9}$$

m is the so called magnetic dipole moment, a generally unknown target specific parameter, and I_3 is the 3×3 identity matrix.

The bandwidth of the signal measured by the magnetometer depends on a number of different factors such as the speed of the passing vehicles and magnetic moment and is typically lower than 100 Hz. Hence, here, the magnetometer is sampled at a sampling rate of 200 Hz. In practice, this sampling rate is supported by the sensors being evaluated for a prototype system [27].

C. Accelerometer

Similarly, the interaction between the vehicle and the road causes road surface waves that can be measured by an accelerometer. It has been shown that the wave propagation can be described as waves spreading circularly from the source [28] with transfer function

$$G(\omega) = A(\omega)\mathcal{H}_0^{(1)}(-k(\omega)|r|)$$
(10)

where $A(\omega)$ is a material parameter and $k(\omega)$ the complex wavenumber. $\mathcal{H}_0^{(1)}(x)$ is the Hankel function of the first kind of order zero [29].

Furthermore, a common approach in tracking is to model the source as a random process [30], which is also used here. Hence, the input is described by the random variable

$$v_t \sim \mathcal{N}(0, 1). \tag{11}$$

Together with a narrow-band approximation and describing each axle of a vehicle as an individual source, the measurement model for a two-axled vehicle becomes (see [31] for a detailed derivation)

$$y_{a,t} = \underbrace{\kappa i \left(\mathcal{H}_0^{(1)}(i\eta | r_{1,t} |) + \mathcal{H}_0^{(1)}(i\eta | r_{2,t} |) \right)}_{\triangleq h_a(x_t)} v_t + w_{a,t}, \quad (12)$$

where $i = \sqrt{-1}$, $\kappa = |A|$ and $\eta = \text{Im}(k)$ are material parameters, which in general are unknown (they may be slowly

varying due to seasonal influences). $w_{a,t}$ is measurement noise with

$$p(w_{a,t}) = \mathcal{N}(w_{a,t}; 0, \sigma_a^2), \tag{13}$$

The location of the two axles is described by the two vectors pointing at the center of each axle given by

$$r_{1,t} = r_t + \begin{bmatrix} l/2 \\ 0 \\ 0 \end{bmatrix}$$
 and $r_{2,t} = r_t - \begin{bmatrix} l/2 \\ 0 \\ 0 \end{bmatrix}$, (14)

and l is the wheelbase which is approximately 2.5 m for passenger cars and is hence assumed known [32].

The bandwidth of the road surface waves depends mainly on the material parameters of the road which in turn can depend on location and climate. It has been indicated that a bandwidth of 1 kHz should be sufficient for year-round operation [28] and hence, the sampling rate for the accelerometer is chosen to be 2 kHz. This is again supported by the sensor under consideration [33].

III. PARTICLE FILTERING

Particle filters are well-suited for state estimation of nonlinear, dynamic systems such as the one introduced in Section II [34], [35]. Here, we first describe the particle filter as a state estimator as known from the literature [34], [35] and then we show a way of extending the filter for using it in multi-rate sensor scenarios.

A. Sampling Importance Resampling Particle Filter

Assume that the dynamic system of interest is described by

$$x_t = f(x_{t-1}, u_t)$$
 (15a)

$$y_t = h(x_t, w_t), \tag{15b}$$

where t = 1, ..., T, x_t is a $d_x \times 1$ state vector, y_t a $d_y \times 1$ measurement vector, u_t and w_t are process and measurement noises, respectively, and the initial state is $x_0 \sim p(x_0)$, with $p(x_0)$ being the prior distribution of x_0 . It is well known that a closed form solution for the filtering density $p(x_t|y_{1:t})$ only exists for special cases of (15), for example when the probability density functions are Gaussian and the system is linear.

For other cases, one has to resort to approximating methodologies. One of them is particle filtering [34], [35], which is also the approach that we adopt in this work. With particle filtering, we approximate the distributions of interests with particles, $x_t^{(m)}$, and their associated weights, $w_t^{(m)}$, where mis the index of a particle and its weight. Each particle can be interpreted as a candidate value of the state of the system while its corresponding weight can be interpreted as the probability of that value. Finally, note that the weights themselves sum up to one.

A particle filtering method has three steps, (a) particle generation (propagation), (b) weight computation, and (c) resampling. The last step is necessary to avoid degeneracy of the discrete representation of the target distribution. With reampling, the particles with low weights are removed and the ones with large weights are replicated. The removal and replication is conducted in random fashion according to the weights of the particles.

There are many different particle filtering methods available in the literature but the original one goes under the name sequential importance resampling (SIR) [36], and it is the method we use in this paper. For completeness, the SIR particle filtering algorithm as it was introduced in [36] is summarized in Algorithm 1.

Algorithm 1 (SIR Particle Filter).

- 1) Set $t \leftarrow 0$ and initialize $x_0^{(m)} \sim p(x_0), w_0^{(m)} = 1/M$ for m = 1, ..., M.
- 2) Set $t \leftarrow t + 1$; continue if $t \le T$, otherwise terminate.
- 3) Propagate the particles,

$$x_t^{(m)} \sim p\left(x_t | x_{t-1}^{(m)}\right)$$

- 4) Calculate the non-normalized weights
 - $\tilde{w}_t^{(m)} = w_t^{(m)} p\left(y_t | x_t^{(m)}\right).$
- 5) Normalize the weights

$$w_t^{(m)} = \frac{\tilde{w}_t^{(m)}}{\sum_{m=1}^M \tilde{w}_t^{(m)}}.$$

- 6) If $\left(\sum_{m=1}^{M} \left(w_t^{(m)}\right)^2\right)^{-1} < M_T$ resample with replacement such that $P(x_t = x_t^{(m)}) = w_t^{(m)}$ and set $w_t^{(m)} = 1/M$.
- 7) Roughen the particles according to (16)-(17).
- 8) Return to step 2.

In this algorithm M is the size of the particle set and M_T is a positive number less than M. In step 6 we decide if resampling should take place or not. If the "effective sample size" of the particle set (expression on the left of the inequality in step 6) is less than the preselected value M_T , resampling is performed. Finally, step 7 is a standard trick to mitigate sample impoverishment [36]. It improves sample diversity by adding an additional jitter to the particles which reduces the effects of the dependency introduced by the resampling step. The roughening is performed by adding an independent jitter c to each particle $x_t^{(m)}$ where

$$c \sim \mathcal{N}(0, \Sigma).$$
 (16)

 Σ is a diagonal covariance matrix with elements

$$[\Sigma]_i = (KEM^{-1/d_x})^2 \tag{17}$$

where K is a tuning parameter and E is the difference between the largest and smallest values of all particles in direction *i*. By choosing the covariance as given in (17), the standard deviation is normalized such that the roughening is proportional (with proportionality K) to the distance E (see [36] for details).

B. Multi-rate SIR Particle Filter

Now, instead of having a single sensor assume that the system is measured by L different sensors at L different sampling rates $f_{s,l}$ for $l = 1, \ldots, L$. Hence, we can not use Algorithm 1 directly and have to extend it in order to accommodate the different sampling rates.

First, assume that the ratios

$$R_l = \frac{f_{s,max}}{f_{s,l}} \tag{18}$$

are integers where $f_{s,max}$ is the highest sampling frequency. (This assumption is made for ease of presentation. Particle filters can handle measurements from sensors acquired with any relationship between the sampling frequencies [37].) The measured signals can then be represented by

$$y_{1,R_{1}t} = h_{1}(x_{R_{1}t}, w_{R_{1}t}),$$

$$y_{2,R_{2}t} = h_{2}(x_{R_{2}t}, w_{R_{2}t}),$$

$$\vdots$$

$$y_{L,R_{L}t} = h_{L}(x_{R_{L}t}, w_{R_{L}t}),$$

(19)

where $R_l t$ denotes that the sample of the *l*th sensor is measured only at time instants $R_l t$ (relative to the fastest sensor which has samples at t = 1, 2, ..., T). Furthermore, a reasonable assumption is that the measurement noises $w_{R_l t}$ and $w_{R_k t}$ are uncorrelated, that is,

$$\mathbf{E}\{w_{R_lt}w_{R_kt}\} = 0 \qquad \text{for} \quad l \neq k. \tag{20}$$

Without loss of generality, assume that we are given the measurements of two sensors (L = 2) with $R_1 = 1$ and $R_2 = 2$. Hence, at t odd, both sensors provide new measurements whereas at t even, only sensor 1 provides new data. Then, at any even time instant t_1 , the posterior can be expressed as

$$\frac{p(x_{t_1}|y_{1,1:t_1}, y_{2,1:2:t_1-1})}{\propto p(y_{1,t_1}|x_{t_1}) \ p(x_{t_1}|y_{1,1:t_1-1}, y_{2,1:2:t_1-1}),}$$
(21)

where \propto stands for "proportional to,"

$$y_{l,1:R_l:t} = \begin{bmatrix} y_{l,1} & y_{l,R_l+1} & \dots & y_{l,t} \end{bmatrix}^\top, \qquad (22)$$

and

$$p(y_{1,t_1}|x_{t_1}) \tag{23}$$

is the likelihood.

On the other hand, for $t_1 + 1$ we have new measurements from both sensors; hence

$$p(x_{t_1+1}|y_{1,1:t_1+1}, y_{2,1:2:t_1+1}) \propto p(y_{1,t_1+1}, y_{2,t_1+1}|x_{t_1+1}) p(x_{t_1+1}|y_{1,1:t_1}, y_{2,1:2:t_1-1}).$$
(24)

Furthermore, given the independence (20) of y_{1,t_1+1} and y_{2,t_1+1} , the likelihood can be factorized as

$$\frac{p(y_{1,t_1+1}, y_{2,t_1+1}|x_{t_1+1}) = p(y_{1,t_1+1}|x_{t_1+1})}{\times p(y_{2,t_1+1}|x_{t_1+1})},$$
(25)

which now directly leads to the multi-rate particle filter as follows.

First, note that the particle weights are calculated using $p(y_t|x_t)$ in step 4 of Algorithm 1. Hence, in the above case,

we would simply use the likelihood (23) for calculating the particle weights when t is even and (25) for t odd.

The generalization to an arbitrary number of sensors is straightforward. At any given time t, sensor l delivers a new measurement if the remainder of the integer division of t-1 and R_l is 0 (note that all the sensors start acquiring measurements at t = 1). That is, if

$$t - 1 - R_l \left\lfloor \frac{t - 1}{R_l} \right\rfloor = 0, \tag{26}$$

the lth sensor will produce a measurement at t. Then, we can define the non-normalized weight $\tilde{w}_{t,l}^{(m)}$ for the particle $x_t^{(m)}$ from the measurement of sensor l as

$$\tilde{w}_{t,l}^{(m)} = \begin{cases} p\left(y_{l,t}|x_t^{(m)}\right), & \text{if } t = R_l \left\lfloor \frac{t-1}{R_l} \right\rfloor + 1\\ 1, & \text{otherwise} \end{cases}$$
(27)

and the overall non-normalized weight $\tilde{w}_t^{(m)}$ becomes

$$\tilde{w}_t^{(m)} = w_{t-1}^{(m)} \prod_{l=1}^L \tilde{w}_{t,l}^{(m)}.$$
(28)

Modifying step 4 in Algorithm 1 with (27)-(28) finally leads to the multi-rate particle filter as summarized in Algorithm 2.

Algorithm 2 (Multi-rate SIR Particle Filter).

- 1) Set $t \leftarrow 0$ and initialize $x_0^{(m)} \sim p(x_0)$, $w_0^{(m)} = 1/M$ for m = 1, ..., M.
- 2) Set $t \leftarrow t + 1$; continue if $t \le T$, otherwise terminate.
- 3) Propagate the particles,

$$x_t^{(m)} \sim p\left(x_t | x_{t-1}^{(m)}\right).$$

- 4) Calculate the non-normalized weights
 - a) For each l = 1,...,L, calculate the individual particle weights w^(m)_{t,l} according to (27).
 b) Calculate the total non-normalized particle weight
 - $\tilde{w}_t^{(m)}$ according to (28).
- 5) Normalize the weights

$$w_t^{(m)} = \frac{\tilde{w}_t^{(m)}}{\sum_{m=1}^M \tilde{w}_t^{(m)}}$$

6) If
$$\left(\sum_{m=1}^{M} \left(w_t^{(m)}\right)^2\right)^{-1} < M_T$$
 resample with replacement such that $P(x_t - x_t^{(m)}) = w_t^{(m)}$ and so

placement such that $P(x_t = x_t^{(m)})$ and set $w_t^{(m)} = 1/M.$

- 7) Roughen the particles according to (16)-(17).
- 8) Return to step 2.

When the measurements of the different sensors are fully asynchronous, they are processed as soon as they arrive. For example, let the latest time instant with a measurement be t_1 and the next measurement be obtained at $t_2 > t_1$. Then one generates the particles for t_2 by

$$x_{t_2}^{(m)} \sim p(x_{t_2} | x_{t_1}^{(m)}) \tag{29}$$

and their non-normalized weights are computed by

$$\tilde{w}_{t_2}^{(m)} = w_{t_1}^{(m)} p(y_{t_2} | x_{t_2}^{(m)}).$$
(30)

IV. PROPOSED METHOD

Given the problem formulation in Section II and the particle filtering algorithms in Section III, it is now shown how the latter is applied for vehicle tracking. In the following, there are two subsections. In the first, we show how to apply the particle filter to the problem as formulated directly, assuming known parameters. In the second, we present how the problem can be dealt with when the parameters of the model are unknown.

A. Known Parameters

Assuming that all the parameters in the measurement models (8) and (12) are known, it is easy to apply the particle filter. The system is completely defined by (1), (8), and (12) and hence, the predictive density and likelihoods are given by

$$p(x_{t+1}|x_t) = \mathcal{N}\left(x_{t+1}; Ax_t, \sigma_{u_t}^2 B B^{\top}\right), \qquad (31a)$$

$$p(y_{m,t}|x_t) = \mathcal{N}\left(y_{m,t}; h_m(x_t), \sigma_m^2 I_3\right), \qquad (31b)$$

$$p(y_{a,t}|x_t) = \mathcal{N}\left(y_{a,t}; 0, h_a(x_t)^2 + \sigma_a^2\right),$$
 (31c)

and thus, applying the particle filter is straightforward.

B. Unknown Parameters

A more realistic scenario dictates that neither the parameters of the measurement function nor the noise characteristics are known. Taking this into account, a straightforward solution would be to also include these parameters in the estimation problem. However, these parameters are often not of interest and what is more, they would unnecessarily also increase the dimension of the problem. Instead, the nuisance parameters can be modeled as unknowns with a prior distribution $p(\theta)$. This has the advantage that one is then able to use Rao-Blackwellization (also called marginalization) in order to integrate out the unknown parameters analytically [38]. Rao-Blackwellization can be thought of as weighted averaging over all the possible values of the parameters, where the probability density function of the parameters is the weighting function. The advantages of this approach are that the dimension of spaces where particles have to be generated is reduced and that the method does not have to deal with constant parameters, which in particle filtering require special treatment. All this entails more accurate and robust tracking. The implementation of the Rao-Blackwellization requires use of hyperparameters for the priors of the unknown parameters. It turns out that that the methods are quite robust against the assumed values of these parameters. The likelihoods quickly overtake the priors defined by these hyperparameters and the posterior accurately points to parts of its support where the true values of the unknown parameters might be. Next, we explain how we implement Rao-Blackwellization on both, the magnetometer and accelerometer measurements.

1) Magnetometer: The 3×1 magnetic moment m in (8) is generally unknown. By assigning a prior distribution to m, it can be marginalized out so that the measurement becomes independent of m. First, note that $h_m(x_t)$ can be rewritten as

$$h_m(x_t) = \frac{3(r_t^{\top}m)r_t - |r_t|^2m}{|r_t|^5} = \underbrace{\frac{3r_tr_t^{\top} - |r_t|^2I_3}{|r_t|^5}}_{\triangleq H_m} m. \quad (32)$$

Second, it is assumed that the prior of m conditioned on σ_m^2 is given by

$$p(m|\sigma_m^2) = \mathcal{N}(m; \mu_m, \sigma_m^2 K_m), \tag{33}$$

which is the conjugate prior for (31b) and K_m is a 3×3 matrix. Then, the marginalized distribution becomes

$$p(y_{m,t}|x_t, \sigma_m^2) = \int_{-\infty}^{\infty} p(y_{m,t}|x_t, \sigma_m^2, m) p(m|\sigma_m^2) \, \mathrm{d}m$$

=
$$\int_{-\infty}^{\infty} \mathcal{N}\left(y_{m,t}; H_m m, \sigma_m^2\right)$$

$$\times \mathcal{N}\left(m; \mu_m, \sigma_m^2 K_m\right) \, \mathrm{d}m$$

=
$$\mathcal{N}\left(y_{m,t}; H_m \mu_m, \sigma_m^2 (I_3 + H_m K_m H_m^\top)\right).$$
 (34)

The last equality in (34) is due to a well-known property of Gaussian random variables [39].

The second unknown is the noise variance σ_m^2 . The inverse Gamma distribution defined as

$$IG(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\beta}{x}}$$
(35)

is known to be the conjugate prior for the unknown variance of a Gaussian random variable [38]. We assign it as a prior to σ_m^2 , that is,

$$p(\sigma_m^2) = IG(\sigma_m^2; \alpha_m, \beta_m).$$
(36)

Upon marginalizing σ_m^2 (see Appendix), we obtain

$$p(y_{m,t}|x_t) = \int_{-\infty}^{\infty} p(y_t|x_t, \sigma_m^2) p(\sigma_m^2) d\sigma_m^2$$

= t $\left(y_{m,t}; 2\alpha_m, H_m \mu_m, \frac{\beta_m (I_3 + H_m K_m H_m^\top)}{\alpha_m}\right),$ (37)

where $t(x; \nu, \mu, C)$ is the multivariate t-distribution with ν degrees of freedom given by

$$t(x;\nu,\mu,C) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\pi^{n/2}\nu^{n/2}\Gamma(\nu/2)|C|^{1/2}} \times \left(1 + \frac{(x-\mu)^{\top}C^{-1}(x-\mu)}{\nu}\right)^{-\frac{\nu+n}{2}},$$
(38)

and where x is of dimension $n \times 1$, μ is the mean of x, and C is an $n \times n$ positive definite scale matrix.

2) Accelerometer: Similar to the magnetometer, the accelerometer measurement model depends on a set of unknown parameters, see (12). Starting from

$$p(y_{a,t}|x_t,\kappa,\eta,v_t,\sigma_a^2) = \mathcal{N}(y_{a,t};h_a(x_t)v_t,\sigma_a^2),$$
(39)

which follows directly from (12), we note that

$$h_{a}(x_{t})v_{t} = \kappa \underbrace{i\left(\mathcal{H}_{0}^{(1)}(i\eta|r_{1,t}|) + \mathcal{H}_{0}^{(1)}(i\eta|r_{2,t}|)\right)}_{\triangleq H_{a}}v_{t} \quad (40)$$

is linear in $\kappa.$ After assigning a conjugate prior conditioned on σ_a^2 and $v_t,$ i.e.,

$$p(\kappa | \sigma_a^2, v_t) = \mathcal{N}\left(\kappa; \mu_\kappa; \frac{k_\kappa \sigma_a^2}{v_t^2}\right),\tag{41}$$

where k_{κ} is a positive hyperparameter, we marginalize with respect to κ and obtain

$$p(y_{a,t}|x_t, \eta, v_t, \sigma_a^2) = \int_{-\infty}^{\infty} p(y_{a,t}|x_t, \kappa, \eta, v_t, \sigma_a^2)$$

$$\times p(\kappa|\sigma_a^2, v_t) d\kappa$$

$$= \int_{-\infty}^{\infty} \mathcal{N}\left(y_{a,t}; \kappa H_a v_t, \sigma_a^2\right)$$

$$\times \mathcal{N}\left(\kappa; \mu_{\kappa}; \frac{k_{\kappa} \sigma_a^2}{v_t^2}\right) d\kappa$$

$$= \mathcal{N}\left(y_{a,t}; \mu_{\kappa} H_a v_t, \sigma_a^2(1 + k_{\kappa} H_a^2)\right).$$
(42)

Next, the unknown measurement noise variance can be marginalized by assuming again an inverse Gamma prior,

$$p(\sigma_a^2) = IG(\sigma_a^2; \alpha_a, \beta_a).$$
(43)

Marginalization then yields a t-distribution, similar to (37), that is,

$$p(y_{a,t}|x_t,\eta,v_t) = \int_{-\infty}^{\infty} p(y_{a,t}|x_t,\eta,v_t,\sigma_a^2)p(\sigma_a^2)d\sigma_a^2$$

= t $\left(y_{a,t}; 2\alpha_a, \mu_\kappa H_a v_t, \frac{\beta_a(1+k_\kappa H_a^2)}{\alpha_a}\right).$ (44)

Furthermore, it is given that $v_t \sim \mathcal{N}(0, 1)$. Unfortunately, marginalization with respect to v_t cannot be used directly in (44). However, $p(y_{a,t}|x_t, \eta, v_t)$ can be approximated by a Gaussian distribution as

$$p(y_{a,t}|x_t,\eta,v_t) \approx \mathcal{N}\left(y_{a,t};\mu_{\kappa}H_a v_t,\beta(1+k_{\kappa}H_a^2)\right), \quad (45)$$

and then, marginalization yields

$$p(y_{a,t}|x_t,\eta) \approx \int_{-\infty}^{\infty} p(y_{a,t}|x_t,\eta,v_t)p(v_t)dv_t$$

=
$$\int_{-\infty}^{\infty} \mathcal{N}\left(y_t^a;\mu_\kappa H_a v_t,\beta(1+k_\kappa H_a^2)\right) \qquad (46)$$
$$\times \mathcal{N}(v_t;0,1)dv_t$$

=
$$\mathcal{N}\left(y_{a,t};0,\beta(1+k_\kappa H_a^2)+\mu_\kappa^2 H_a^2\right).$$

The last unknown parameter η appears as an argument to the Hankel function $\mathcal{H}_0^{(1)}(x)$ in (12). It is thus impossible to analytically marginalize with respect to it and hence, it has to be taken care of in a different way. One straightforward approach is to augment the state vector with the unknown parameter as

$$\tilde{x}_t = \begin{bmatrix} x_t \\ \eta \end{bmatrix} \tag{47}$$

which leads to the augmented process

$$\tilde{x}_{t} = \underbrace{\begin{bmatrix} 1 & T_{s} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq \tilde{A}} \tilde{x}_{t-1} + \underbrace{\begin{bmatrix} \frac{T_{s}^{2}}{2} \\ T_{s} \\ 0 \end{bmatrix}}_{\triangleq \tilde{B}} u_{t}$$
(48)

with prior

$$p(\eta) = \mathcal{N}(\eta; \mu_{\eta}, \sigma_{\eta}^2) \tag{49}$$

which is used to propose samples at t = 1.

3) Summary: Finally, the system where the unknown parameters have been addressed is defined by (37), (46), and (48). This leads to the predictive density and likelihoods defined by

$$p(\tilde{x}_{t+1}|\tilde{x}_t) = \mathcal{N}\left(\tilde{x}_{t+1}; \tilde{A}\tilde{x}_t, \sigma_{u_t}^2 \tilde{B}\tilde{B}^\top\right),$$
(50a)

$$p(y_{m,t}|\tilde{x}_t) = t\left(y_{m,t}; 2\alpha_m, H_m\mu_m, \frac{\beta_m(I_3 + H_mK_mH_m^{\scriptscriptstyle \top})}{\alpha_m}\right)$$
(50b)

$$p(y_{a,t}|\tilde{x}_t) \approx \mathcal{N}\left(y_{a,t}; 0, \beta_a(1+k_\kappa H_a^2) + \mu_\kappa^2 H_a^2\right), \qquad (50c)$$

where α_m , β_m , K_m , β_a , k_κ , and μ_κ are hyperparameters.

V. NUMERICAL ILLUSTRATION

In this section we demonstrate the performance of the proposed tracking method by computer simulations using the models introduced in Section II. We considered five different scenarios. First, the multi-rate particle filter was run with known parameters in order to show that the approach is feasible. Second, the marginalized particle filter was simulated and its performance analyzed. Third, two consecutively passing vehicles, one fast and one slower, were simulated. Fourth, we analyze the sensitivity of the proposed method to hyperparameters. Fifth, we compared the method of fusing the sensors using particle filtering to particle filters for the individual sensors only, a particle filter fusing both sensors but not using multi-rate processing, and the standard unscented Kalman filter (UKF) [40]. In each setup, a total of 100 Monte Carlo (MC) simulations were executed to evaluate the average performance.

The hyperparameters for the magnetometer were set as

$$\mu_m = \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \quad \text{and} \quad K_m = I_3,$$

and the parameters of the magnetometer measurement noise were

 $\sigma_{m,0}^2 = 1 \times 10^{-5}, \quad \alpha_m = 6, \quad \text{and} \quad \beta_m = \sigma_{m,0}^2 (\alpha_m - 1).$

Furthermore, for the constanst κ and η of the accelerometer, the hyperparameters were

$$\mu_{\kappa} = 100, \text{ and } k_{\kappa} = 1$$

and

$$\mu_{\eta} = 4$$
, and $\sigma_{\eta}^2 = 1$,

respectively. For the accelerometer measurement noise, we used

$$\sigma_{a,0}^2 = 1 \times 10^{-8}, \quad \alpha_a = 6, \text{ and } \beta_a = \sigma_{a,0}^2(\alpha_a - 1).$$

The choice of these parameters yields typical signal- and noise levels encountered in commercially available sensors.

In all the cases, the initial distribution for the particles was chosen as Gaussian with mean and covariance given by

$$\mu_{x_0} = \begin{bmatrix} -10\\15 \end{bmatrix} \quad \text{and} \quad C_{x_0} = \begin{bmatrix} 10 & 0\\0 & 100 \end{bmatrix}$$

In all the simulations, we used M = 1,000 particles.



Fig. 1. Simulated measurement signals. (a) Magnetometer signal and (b) accelerometer signal.

A. Known Parameters

First, we show how the multi-rate particle filter performs when all the model parameters are known, that is, by using the model summarized in (31). Samples of simulated measurement signals of the magnetometer and accelerometer are displayed in Fig. 1. We obtained all the results with the assumption that the vehicle started at $r_0^x = -5$ m with $\dot{r}_0^x = 20$ m/s.

In the case of known parameters, we expected good performance because there were no uncertainties about them. Figure 2 shows the mean error of the estimated states from the 100 simulations (solid line). We see that both states converge quickly from the initial offset (due to the initialization) toward zero. Further, the 2σ -bounds (marked with dotted lines) indicate that even the uncertainty about the estimated states converges quickly as the vehicle approaches and passes the sensor. Finally, it is also worth pointing out the initial behavior of the filter (up to $t \approx 200$). While the position error converged almost immediately, the speed estimation error converged somewhat more slowly as the target approached the sensor which is due to the fact that the speed is not observed through the measurement directly.

B. Unknown Parameters

In Fig. 3, we show the results for the same scenario as in the previous section, however in this case the parameters are assumed unknown. The results were obtained by using the model (50). The mean of the state estimation error in Fig. 3 suggests that even in this case, both the position and speed error converged towards zero for the multi-rate particle filter. However, note that the uncertainty (dotted line) is larger as compared to the the case where the parameters were known. This can be attributed to the increased uncertainty in the likelihood caused by the marginalization.



Fig. 2. Mean estimation error with known parameters. (a) Position and (b) speed.



Fig. 3. Mean estimation error with unknown parameters. (a) Position and (b) speed.

C. Two Vehicles

Fig. 4 illustrates the measurement signal for two vehicles passing by the sensors one after another. The first vehicle passed the sensor with an initial speed of $\dot{r}_0^x = 20 \text{ m/s}$ while the second vehicle passed with $\dot{r}_0^x = 10 \text{ m/s}$. The two vehicles started at $r_0^x = -5 \text{ m}$. The particle filter was initialized whenever a vehicle was detected in front of the sensor using the magnetometer (vehicle detection using magnetometers has been shown to be reliable and rather simple, see, for example [41]). Again, 100 MC simulations were run.

The results of the mean estimation error are presented in



Fig. 4. Simulated measurement signals for two consecutive vehicles. (a) Magnetometer signal and (b) accelerometer signal.



Fig. 5. Mean estimation error for two vehicles following each other. (a) Position and (b) speed.

Fig. 5 (solid lines) together with the 2σ -bounds (dotted lines). First, note how the particle filter was only activated whenever a vehicle was present. Here this was the case from $t \approx 100$ to $t \approx 900$ for the first vehicle and from $t \approx 1,500$ to $t \approx 2,600$ for the second vehicle. For both vehicles, the position error of the multi-rate filter converged to zero very quickly (Fig. 5a), whereas the speed error converged more slowly (Fig. 5b), as was the case in the experiment with a single vehicle. This is especially pronounced for the second, slower vehicle.

 TABLE I

 PARAMETERS FOR THE HYPERPARAMETER SENSITIVITY ANALYSIS.



Fig. 6. Examples of the proposed method using different hyperparameters. (a) Position and (b) speed.

D. Sensitivity to Hyperparameters

The proposed method heavily relies on Rao-Blackwellization, which involves analytical integration of some of the unknown parameters. Rao-Blackwellization requires the use of prior distributions of the parameters to be integrated, and these priors are in turn defined by their own parameters (hyperparameters). Hence, in this section, the sensitivity of the method to the hyperparameters introduced through the priors is illustrated. For this purpose, a single vehicle passage with varying true- and hyperparameters is considered. The combinations of the different parameters are given in Table I.

The results are shown in Fig. 6. If the location hyperparameter μ_m is close to the true value of m, the performance does not vary much depending on the prior variance-related parameter K_m , see cases a) and b). However, it can be seen that choosing μ_m significantly different from the true value of m can lead to deteriorated performance, see case c). This effect can be mitigated by instead increasing the hyperparameter K_m and thereby increasing the prior variance (case d), see (33)). We note that an increased prior variance entails less reliance on the prior and more on the likelihood, which is why the results of case d) are better than the ones of case c).



Fig. 7. Comparison of the multi-rate particle filter and individual particle filters for each sensor. RMSE for (a) the position and (b) the speed.

E. Comparison to Other Filters

In order to further illustrate the performance of the proposed method, it is compared to individual particle filters for each of the sensors first. Again, a single passage of a vehicle and 100 Monte Carlo simulations using the setup mentioned above are considered. As a measure of comparison, the root mean square error (RMSE) of the individual states over the 100 simulations was chosen.

Figure 7 illustrates the obtained results. It can be seen that the filter fusing both measurements (solid line) has the lowest RMSE for both states, the position (Fig. 7a) and speed (Fig. 7b). The filter only using the magnetometer (dashed line) performs as well as the fusion filter initially but does not attain as low errors once the vehicle is in front of the sensor around $t \approx 500$. Finally, the filter only using the accelerometer (dotted line) performs considerably worse than both other filters.

As mentioned above, popular alternatives to state estimation in nonlinear systems using particle filtering includes different non-linear Kalman filters such as the extended Kalman filter (EKF) or the UKF [40]. It is generally believed, that the UKF is superior to the EKF and hence, the particle filter is compared to the UKF here. For comparison, we employed an UKF that uses multi-rate fusion of the data just as the proposed particle filter. This is achieved by calculating the time and measurement updates by simply using all the available measurements for any given t. Additionally, the proposed method is also compared to a particle filter fusing the two sensors where both sensors are sampled at the same sampling frequency.

The results of this comparison are shown in Fig. 8. Even though the UKF (dashed) initially converges, it has difficulties to actually track the vehicle which is reflected in increased RMSE around $t \approx 250$ for both, the position and speed. The



Fig. 8. Comparison of fusion using the multi-rate particle filter, a regular particle filter, and an unscented Kalman filter. RMSE for (a) the position and (b) the speed.

time at which this happens coincides with the time where the vehicle enters the range of the sensor (see Fig. 1). Closer inspection of the results revealed that the UKF is not able to run on the accelerometer measurements only (in between the magnetometer sampling instants) which eventually leads to divergence. By contrast, both particle filters are able to do it, and therefore, they clearly outperform the UKF. Hence, the additional complexity of the particle filter results in a gain in performance. Comparing the multi-rate particle filter to the regular particle filter (dotted), we can see that the regular particle filter performs better. This is not surprising since it can make use of data which is unavailable to the multi-rate particle filter.

F. Discussion

The above simulations show that the multi-rate particle filter performed well. It successfully tracks the vehicle with low uncertainty. As expected, it yielded best results because it processes information from the two sensors in comparison to the particle filters that rely on single sensors only. The reason this is not a surprising result is because the statistical information from the sensors is additive, that is, more measurements lead to more accurate estimates.

Furthermore, the simulations indicate that even though some parameters of the model are unknown, they can be treated by Rao-Blackwellization. The results suggest that the error for the proposed multi-rate algorithm was of the same magnitude as in the case where the parameters were assumed known. It was also shown that the choice of the hyperparameters affect the method in the sense that poorly chosen parameters can lead to deteriorated performance. In practice, one often has insight in the system and can calibrate the parameters accordingly. This can, for example, be achieved by gathering enough measurement data and building the appropriate prior and using empirical Bayes methods, see [42].

In summary, it is apparent that tracking should be done by using both sensors. It is important to point out that the multi-rate particle filter does not require equal sampling rates or synchronized sampling by the two sensors. These sensors can provide samples according to their sampling rates. This, in general, allows for conservation of energy and reduction of necessary computations.

It is also worth mentioning scenarios where the proposed method has difficulties to accurately track vehicles. This mostly happens when the single-target assumption is violated. Typical examples when this occurs are vehicles crossing in front of the sensor or vehicles following each other very closely. In order to accommodate such scenarios, the target model has to be extended to account for the additional vehicle.

Finally, note that the simulations are conducted using the models introduced in Section II. Since there is no model mismatch between the developed estimator and the simulation setup, the numerical illustrations do not provide inaccuracies due to any mismatches between reality and the assumed models. We would like to point out that the model for the magnetometer measurements has been verified (see, for example, [15]) whereas the one for accelerometer measurements is still under way. If it turns out that the accelerometer model should be significantly different, the particle filtering method proposed in this paper can be straightforwardly modified.

VI. CONCLUSION

In this paper we addressed the problem of vehicle tracking based on multi-rate particle filtering that fuses measurements from two different sensors. The sensors are an accelerometer and a magnetometer and they operate with different sampling rates. The measurement models include unknown parameters, and we handle them by using Rao-Blackwellization. The results indicate that the proposed approach is feasible. As expected, better results for the state estimates were obtained when the multi-rate particle filter used the measurements of both sensors as opposed to using measurements of each sensor individually. Furthermore, the performance of the filter when the parameters of the models were assumed unknown was very close to that of the filter that knew the correct values of the parameters.

APPENDIX

In order to find the marginalized distribution for $p(y_{m,t}|x_t)$, where σ_m^2 has been eliminated, we start from the likelihood (34) given by

$$p(y_{m,t}|x_t,\sigma_m^2) = \mathcal{N}\left(y_{m,t}; H_m\mu_m, \sigma_m^2(I_3 + H_mK_mH_m^\top)\right),$$

and the prior distribution

$$p(\sigma_m^2) = \frac{\beta_m^{\alpha_m}}{\Gamma(\alpha_m)} \left(\sigma_m^2\right)^{-\alpha_m - 1} e^{-\frac{\beta_m}{\sigma_m^2}}$$

Then, the marginalized likelihood is given by

$$\begin{split} p(y_{m,t}|x_t) &= \int_{-\infty}^{\infty} p(y_{m,t}|x_t, \sigma_m^2) p(\sigma_m^2) \mathrm{d}\sigma_m^2 \\ &= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2} (\sigma_m^2)^{3/2} |I_3 + H_m K_m H_m^\top|^{1/2}} \\ &\times e^{-\frac{1}{2\sigma_m^2} (y_{m,t} - H_m \mu_m)^\top (I_3 + H_m K_m H_m^\top)^{-1} (y_{m,t} - H_m \mu_m)} \\ &\times \frac{\beta_m^{\alpha_m}}{\Gamma(\alpha_m)} \left(\sigma_m^2\right)^{-\alpha_m - 1} e^{-\frac{\beta_m}{\sigma_m^2}} \mathrm{d}\sigma_m^2 \\ &= \frac{\beta_m^{\alpha_m}}{(2\pi)^{3/2} \Gamma(\alpha_m) |I_3 + H_m K_m H_m^\top|^{1/2}} \\ &\times \int_0^{\infty} \frac{1}{(\sigma_m^2)^{\frac{2\alpha_m + 5}{2}}} e^{-\frac{\kappa_m^2}{\sigma_m^2}} \mathrm{d}\sigma_m^2, \end{split}$$

where

$$K = (y_{m,t} - H_m \mu_m)^\top (I_3 + H_m K_m H_m^\top)^{-1} (y_{m,t} - H_m \mu_m) + \beta$$

Using integration by substitution with

$$\begin{split} \gamma &= \frac{K}{\sigma_m^2} \\ d\sigma_m^2 &= -\frac{(\sigma_m^2)^2}{K} d\gamma \end{split}$$

yields for the integral

$$\begin{split} &\int_{0}^{\infty} \frac{1}{(\sigma_{m}^{2})^{\frac{2\alpha_{m}+5}{2}}} e^{-\frac{K}{\sigma_{m}^{2}}} d\sigma_{m}^{2} = \int_{0}^{\infty} \frac{1}{K(\sigma_{m}^{2})^{\frac{2\alpha_{m}+1}{2}}} e^{-\gamma} d\gamma \\ &= \frac{1}{K} \int_{0}^{\infty} \left(\frac{K}{K}\right)^{\frac{2\alpha_{m}+1}{2}} \frac{1}{(\sigma_{m}^{2})^{\frac{2\alpha_{m}+1}{2}}} e^{-\gamma} d\gamma \\ &= \frac{1}{K^{\frac{2\alpha_{m}+3}{2}}} \int_{0}^{\infty} \left(\frac{K}{\sigma_{m}^{2}}\right)^{\frac{2\alpha_{m}+1}{2}} e^{-\gamma} d\gamma \\ &= K^{-\frac{2\alpha_{m}+3}{2}} \int_{0}^{\infty} \gamma^{\frac{2\alpha_{m}+3}{2}} e^{-\gamma} d\gamma \\ &= K^{-\frac{2\alpha_{m}+3}{2}} \Gamma\left(\frac{2\alpha_{m}+3}{2}\right). \end{split}$$

By resubstituting K and rearranging, we finally obtain

$$\begin{split} p(y_{m,t}|x_t) &= \frac{\Gamma\left(\frac{2\alpha_m + 3}{2}\right)}{(2\pi)^{3/2}\beta^{3/2}\Gamma(\alpha_m)|I_3 + H_m K_m H_m^\top|^{1/2}} \\ &\times \left(\frac{(y - H_m \mu_m)^\top (I_3 + H_m K_m H_m^\top)^{-1} (y - H_m \mu_m)}{2\beta} \right. \\ &+ 1 \right)^{-\frac{2\alpha_m + 3}{2}} \\ &= t \left(y_{m,t}; 2\alpha_m, H_m \mu_m, \frac{\beta_m (I_3 + H_m K_m H_m^\top)}{\alpha_m}\right), \end{split}$$

....

. . .

which is the multivariate t-distribution with shape parameter $2\alpha_m$, location $H_m\mu_m$, and scale matrix $\beta_m(I_3 + H_m K_m H_m^{\top})/\alpha_m$.

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